Theoretical and Computational Aspects of Ramsey Theory

Examensarbete i matematik fördjupningskurs 20 poäng

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Abstract

Ramsey theory is an area of combinatorics which is concerned with how large structures can become without containing various substructures. In this paper Ramsey theory is discussed in the context of graph theory, which is one of the more common ways of looking at it. We will discuss classical two-color graph Ramsey numbers, i.e. the smallest values of n for which a complete graph on n points, where every edge is colored either red or blue, must contain either a blue K_x or a red K_y. Various known theorems for bounds on these numbers are discussed, and then implemented in the programming language MATLAB. By using these routines we have been able to duplicate many earlier calculations which various people have done by hand, and have also been able to improve on one published bound: $R(3,15) \leq 89$ is sharpened to 88. We also give a proof that $R(3,12) \leq 59$ (the best previously published bound is 60) by examining possible graph structures.

Contents

1. Introduction: What is Ramsey Theory?

Suppose that you are asked the following question: How many people must there be at a party so that we can be absolutely certain that we have among them a group of three people who either all know each other, or none of them know each other?

A useful mathematical tool for this type of problem (as well as many others) is the branch of combinatorics known as *graph theory*. A graph in the context of combinatorics can be thought of as a set of objects, which we will call vertices, together with a set of relationships between them. There is a natural way to visualize graphs: we can depict the vertices as points and the relationships as edges between the points. If we are interested in all the possible relations between a given number, say n, of objects, we can depict this situation using a *complete graph*: n points with all possible edges drawn in. Figure 1 shows some complete graphs.

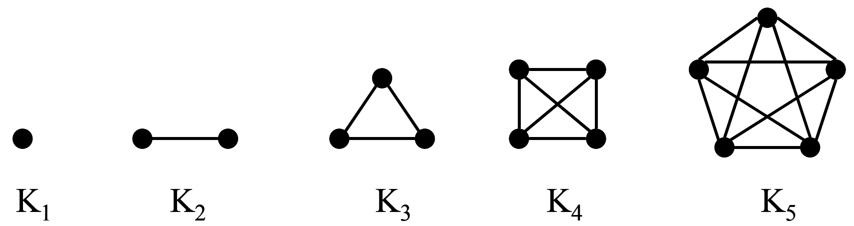

Figure 1. Some complete graphs K_n

To use graph theory to solve the above problem, we can think of each complete graph on n vertices as representing n people, and the edge between any two people as representing the relationship between them. Now, we can color each edge of the graph blue, if the two people joined by this edge know each other, or red if they do not know each other. Our question above now becomes: How many vertices must we have in a complete graph where every edge is either red or blue, before we are certain that there somewhere in this graph exists either a red triangle or a blue triangle?

Obviously, if we have three vertices it is possible to have a triangle, but it is of course not necessary: We might have persons A and B knowing each other, say, and both not knowing C.

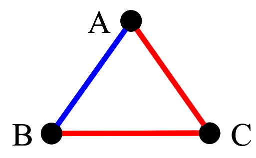

Figure 2. A coloring of K_3 with no monochromatic triangle

Next we want to examine the complete graph on four vertices: Is it possible to color the edges of this graph so that there is neither a red triangle nor a blue one? This is not hard to do, figure 3 shows one possible solution.

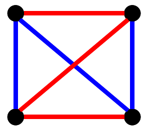

FIGURE 3. A coloring of K_4 with no monochromatic triangles

Five vertices require slightly more thought, but it is still possible to find a coloring of this graph which has no triangles:

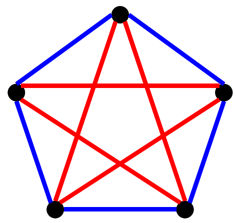

FIGURE 4. A coloring of K_5 with no monochromatic triangles

If we try to find such a coloring for the complete graph on six vertices, we run into trouble. Using trial and error it seems impossible to find a triangle-free coloring, but to verify that no such coloring exists we need a proof. Here is one way to do it:

Choose one of the vertices and label it v. Notice that in a complete graph on six vertices, each vertex has five edges adjacent to it. No matter how we color the five edges which are adjacent to v, we can be certain that at least three of them have the same color. We can assume these three edges are all blue, and label the three vertices adjacent to them a, b, and c. Next we examine the edges ab, bc and ca: If any one of these three edges is blue (for example ab), then that edge forms a blue triangle with the two edges adjacent to v (va and vb). But if none of ab, bc and ca are blue, then all three must be red and we have a red triangle. Of course the same argument can be applied if our three original edges are all red.

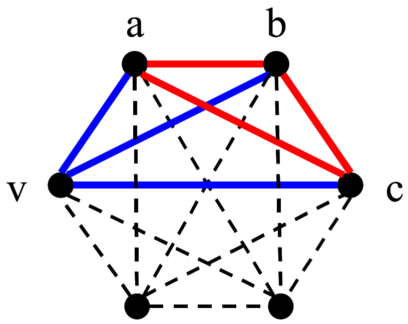

FIGURE 5. A proof that any coloring of K_6 must contain at least one monochromatic triangle

Thus we have proved that it is impossible to have a coloring of the complete graph on six vertices that does not somewhere include either a red triangle or a blue triangle, or in the context of people in a room that we discussed earlier that if we have six people in a room we can be certain that there is a trio where either all three know each other or all are strangers.

Problems like this are the basis of the branch of mathematics known as Ramsey theory. The basic question asked is usually "How large a structure can we have without being certain that it contains a certain substructure?". One of the main theorems of Ramsey theory can be stated (somewhat informally) as "Total disorder is impossible", or in other words: If we have a randomly (dis-) organized structure we can be sure that some small part of it is ordered. One of the usual ways of looking at Ramsey theory is in the context of graph theory. We examine colorings of graphs and try to determine which conditions that apply; how large the graphs can be without containing certain substructures. In this paper we will concern ourselves mainly with the following question: If we have a complete graph G on n vertices where every edge is colored either blue or red, what is the smallest value of n that guarantees the existence of either a complete graph on x vertices which is blue, or a complete graph on y vertices which is red?

In the example above the question was what number of vertices is required in a two-color complete graph to guarantee the existence of a triangle (which is a complete graph on three vertices) which is either red or blue. In the terminology of Ramsey theory, this question is "What is the value of $R(3,3)$, the so-called Ramsey number?" As we saw, the answer to this question is 6. The obvious next question is now: if $R(3,3)$ is 6, what is for example $R(3,4)$? (In other words, how large a complete graph must we have to guarantee the existence of either a red triangle or a complete graph on 4 vertices?) We can of course continue in this way: What are $R(4,4)$, $R(4,5)$, $R(5,5)$ and so on? It turns out that the answers to these questions and other similar ones are far from obvious; so far no one has found a general procedure for finding solutions to problems of this sort aside from programming a computer to test all possible combinations, and the number of computations involved grows so quickly that to compute $R(5,5)$ by this method would probably take several hundred years of computer time.

The rest of this paper will be devoted to examining the techniques in this field which have been developed by various people over the years, and to an implementation of these techniques in the programming language MATLAB. We have written a package of MATLAB functions, called FRANK after Frank Ramsey, who originated Ramsey Theory in 1930. (Or for those who think that computer programs should have acronyms as names, First Ramsey ANalysis Kit). One of the most promising techniques is to examine the number of edges possible in a graph which satisfies the desired conditions: If we examine the possible colorings of the graph on 4 vertices which is shown in figure 3, we can see that we do not have to have exactly three edges of each color to ensure that there are no triangles in either color. In fact, it is possible to have two, three or four edges which are the same color and still have no triangle (see figure 8 in section 4.3) but five edges of the same color would mean that there has to be a triangle. Many interesting conclusions can be drawn by using these edge number values, (called e) and so a large part of FRANK is devoted to calculation of these e-numbers.

2. Terminology and basic definitions

The following definitions will be used throughout:

A *graph* $G = (V, E)$ is formally a set V of vertices together with a set E of edges, which are unordered pairs from the set V. All graphs considered in this paper are simple (any two vertices are connected by at most one edge), undirected (an edge from a to b is considered to be equivalent to an edge from b to a) and free of loops (there can be no edge from a vertex to itself).

A *subgraph* H of a graph G is a subset of the vertices in G together with all the edges from G between these vertices. (Note that this is often elsewhere known as an induced subgraph)

The *complement* of a graph $G = (V, E)$ is designated $\overline{G}$ and is defined as the graph $(V, K_n \setminus E)$ or in other words is the graph on the same vertices as G with all the possible edges between them which are not in G.

A *walk* in a graph G is a sequence of vertices $v_1, v_2, v_3...$ where there exists an edge between each two consecutive vertices.

A *path* in a graph G is a walk where no vertex is repeated. Paths will be designated by the number of vertices they contain: i.e. P_3 is the path which has three vertices and two edges. A path from v to v (with no other repeated vertices) is called a *cycle*, and a *g-cycle* is a cycle with g vertices..

The *girth* of G is the least value of g for which G contains a g-cycle or ∞ for a graph with no cycles.

The *link* of a vertex, $lk(v)$, is the set of edges adjacent to v.

The *degree* or *valence* of a vertex, $\deg(v)$, is the number of edges adjacent to v, or in other words $\deg(v) = |lk(v)|$.

The *second degree of a vertex*, $\deg_2(v)$, is the sum of the degrees of the vertices adjacent to v.

A *k-regular graph* is a graph where the degree of every vertex is exactly k.

The *complete graph* K_n is the graph on n vertices with all $\binom{n}{2}$ possible edges. (Note that K_n is thus $(n-1)$-regular)

The *bipartite complete graph* $K_{m,n}$ is the graph on $m+n$ vertices where each of the m vertices has an edge to each of the n vertices, but there are no other edges. The *multipartite complete graph* $K_{m,n,o...}$is defined analogously.

An *n-coloring* of a graph G is formally a partition of the edges of G into n classes (colors).

The *independence number* $I(G)$ of a graph G is defined as the largest possible number of vertices which can be selected in G so that no two are adjacent to the same edge.

The *clique number* $C(G)$ is defined as the number of vertices in the largest possible complete subgraph of G.

G is called an *(x, y)-graph* if $x > C(G)$ and $y > I(G)$.

The number $e(x, y, n)$ is defined as the minimum number of edges possible in an (x, y)-graph on n vertices. If there are no possible (x, y)-graphs on n vertices, $e(x, y, n)$ is defined to be ∞.

An (x, y)-graph with n vertices and e edges is often called an *(x, y, n)-graph* or an *(x, y, n, e)-graph.*

3. Summary of known results

The two-color Ramsey number $R(x, y)$ is defined as the smallest number n such that a red-blue coloring of the complete graph K_n must contain either a blue K_x or a red K_y. This can also be expressed in terms of independence and clique numbers for the subgraph which consists of all the blue edges in such a coloring: $R(x, y)$ is the smallest number n such that there is no (x, y)-graph on n vertices, or in other words, all graphs with n or more vertices must contain either a clique on x vertices or an independent set on y vertices (corresponding to a red K_y in the two-colored graph.).

It is easy to see that R is symmetric in x and y by simply interchanging the red and blue edges, so we have $R(x, y) = R(y, x)$. This also implies that if we have an (x, y)-graph G, then $\overline{G}$ must be a (y, x)-graph.

The existence of the number $R(x, y)$ for all x and y was proved by Ramsey in 1930. His original proof, as well as a modern rephrasing of it, are given in [2]. Theorem 2 is a special case of Ramsey's theorem.

A partial table of known values and bounds for $R(x, y)$ is given below:

y \ x	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6_1	9_1	14_2	18_3	23_3	28_4	36_5	40 43_6	46 51_7	52 $\mathbf{59}_8$	59 69_7	66 78_7	73 $\mathbf{88}_9$
4		18	25	35 41	49 61	55 84	69 115	80 149	96 191	128 238	131 291	136 349	145 417
5			43 49	58 87	80 143	95 216	116 316	141 442	153	181	193	221	237

y \ x	16	17	18	19	20	21
3	79 $\mathbf{99}_{10}$	92 $\mathbf{110}_{10}$	98 $\mathbf{121}_{10}$	106 $\mathbf{133}_{10}$	109 $\mathbf{145}_{10}$	122 $\mathbf{158}_{10}$

These values, with the exception of the boldface values, are taken from reference [6]. A complete list of references for the above table is given there. For many values, it is necessary to calculate values of $e(x, y, n)$. A table of these values is given in appendix B.

1. These two values follow immediately from the upper- and lower-bound inequalities, theorems 4 and 5.

2. This value can be calculated using the computational system FRANK. It is a fairly simple consequence of recursive formulas for bounds on the number of edges in an (x, y)-graph.

3. Graver and Yackel proved these by hand in [3].

4. McKay and Ke Min proved $R(3, 8)$ to be exactly 28 in [5], by using computer algorithms to eliminate all possible $(3, 8, 28)$-graphs.

5. Grinstead and Roberts originally proved this in [4] by using computer algorithms. We have been able to achieve the same bound using FRANK with edge-number values from [3], [4], and [7] as input.

6. This is also proved in [4].

7. These bounds are proved by Radziszowski and Kreher in [7]. We have been able to calculate the same values by using FRANK and several edge-number values from [1], [3], [4] as well as some of the edge-number values given in [7].

8. This bound is given as 60 in [6]. In chapter 5 we prove that this can be sharpened to 59 by examining possible cases..

9. This bound is given as 89 in [6], but using FRANK we have sharpened it to 88 by using edge-number values calculated in [1], [3], [4], [5] and [7].

10. These are the upper bounds calculated by FRANK, using all available results mentioned above. We have not seen other published values for $x \geq 16$.

4. Theorems and calculational methods

4.1. **Some useful notation and definitions.** A very useful idea in this area is the idea of preferring a vertex, and thus decomposing a graph G into three distinct subgraphs. This is done in the following way: For any graph G, let v be a vertex in G. Now consider all the vertices of G which are joined to v by an edge. These vertices, together with any edges joining them, form a subgraph of G which we will call $H_1(v)$. The vertices different from v and not contained in $H_1(v)$ span another graph which we will call $H_2(v)$. So we now have three distinct subgraphs: the vertex v, $H_1(v)$ and $H_2(v)$. The graphs $H_1(v)$ and $H_2(v)$ will sometimes be denoted simply by H_1 and H_2, or, if there are several graphs under consideration, by $H_1(G,v)$ and $H_2\,(G,v)$. Note that if the graph G contains no triangles, then for any v $H_1(v)$ consists only of disjoint vertices, since an edge joining any two of the vertices in $H_1(v)$ would imply that these two vertices formed a triangle with the vertex v in G.

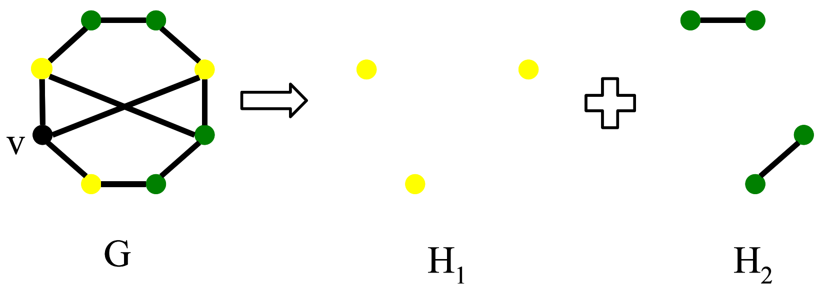

Figure 6. Decomposition of a graph G into $H_1(v)$ and $H_2(v)$

Lemma 1. *If G is an (x,y)-graph and v is any vertex in G, then $H_1(v)$ is an $(x-1,y)$-graph and $H_2(v)$ is an $(x,y-1)$-graph.*

Proof. For any vertex v in G, let H_1 denote the subgraph of vertices joined to v as defined above. Now any set of independent vertices in H_1 must also be independent in G, since there are no more edges between these vertices as seen as vertices in G than there are in H_1. Since we know that $I(G) < y$ because G is an (x,y)-graph, we have $I(H_1) \leq I(G) < y$. If we consider the clique numbers instead we have the following relationship: If H_1 contains a complete subgraph on k vertices, then the graph spanned by these vertices and the preferred vertex v is a complete subgraph on $k+1$ vertices, since v has an edge to every vertex in H_1. Thus $C(H_1) \leq C(G) - 1 < x - 1$, so H_1 is an $(x-1, y)$-graph.

To show that H_2 is an $(x, y-1)$-graph, first note that the clique number of H_2 cannot be greater than the clique number of G, and then that any independent set in G of size y becomes an independent set of size $y-1$ in H_2, since we have removed the vertex v. ∎

4.2. **Simple upper and lower bounds for Ramsey numbers.**

Theorem 1. $R(2, y) = y$

Proof. A $(2, y)$-graph is a graph that contains no 2-clique, and since a 2-clique in a graph is an edge, this means that a $(2, y)$-graph consists completely of disjoint vertices. As soon as we have y of these we have an independent y-set, so $R(2, y) = y$. ∎

Corollary 1. $R(x, 2) = x$ *(since R is symmetric)*

Theorem 2. $R(x, y) \leq R(x, y-1) + R(x-1, y)$ *, the strict inequality holding if* $R(x, y-1)$ *and* $R(x-1, y)$ *are both even.*

This theorem is a special case of Ramsey's theorem. To prove it we will first prove a lemma.

Lemma 2. *If G is an (x, y)-graph on n vertices then the maximum possible valence for any vertex in G is $R(x-1, y) - 1$, and the minimum possible valence is $n - R(x, y-1)$.*

Proof. Since H_1 is an $(x-1, y)$-graph, we must have $\deg(v) < R(x-1, y)$ for any vertex v in G. Since H_2 is an $(x, y-1)$-graph we must have $\overline{deg}(v)$, the valence of v in $\overline{G}$, $< R(x, y-1)$. But $\deg(v) + \overline{deg}(v)$ must equal $n-1$, since for each of the n vertices in G except v itself there either is an edge between them in G or in $\overline{G}$. Hence $\deg(v) > (n-1) - R(x, y-1)$. ∎

Proof. Now we are ready to prove our theorem: The first inequality follows from the fact that the minimum valence in any graph must be less than or equal to the maximum valence:

$n - R(x, y-1) \leq R(x-1, y) - 1$

$n \leq R(x-1, y) + R(x, y-1) - 1$

we observe that the existence of an (x, y) graph on n vertices implies that $n \leq R(x, y) - 1$:

$R(x, y) - 1 \leq R(x-1, y) + R(x, y-1) - 1.$

Now assume equality when $R(x, y-1)$ and $R(x-1, y)$ are both even. In this case there must exist a graph G with $n = R(x, y-1) + R(x-1, y) - 1$ vertices, and n must be an odd number.

Now from the above lemma we have that for every vertex v in G:

$\deg(v) \geq n - R(x, y-1) = R(x, y-1) + R(x-1, y) - 1 - R(x, y-1) = R(x-1, y) - 1$

and also that

$\deg(v) \leq R(x-1, y) - 1$

hence every vertex has valence exactly $R(x-1, y) - 1$, which must be an odd number since $R(x-1, y)$ was even. But this would imply that G has an odd number of points of odd valence, which is impossible. ∎

$n - R(x, y-1) \leq R(x-1, y) - 1$
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hence every vertex has valence exactly $R(x-1, y) - 1$, which must be an odd number since $R(x-1, y)$ was even. But this would imply that G has an odd number of points of odd valence, which is impossible.

Theorem 3. $R(x, y) \geq R(x, k) + R(x, y-k+1) - 1$

Proof. We consider the disjoint union G of two graphs G_1 and G_2.The clique number of G must be equal to the larger of the clique number of G_1 and the clique number of G_2, since no new cliques can be formed by taking the disjoint union. The independence number of G, on the other hand, must be equal to the sum of the two independence numbers. Now let G_1 be an (x, k)-graph and G_2 be an $(x, y-k+1)$-graph. Since an (x, k)-graph can have up to $R(x, k) - 1$ vertices from the definition of $R(x, k)$, we can see that G can have up to $R(x, k) - 1 + R(x, y-k+1) - 1 = R(x, k) + R(x, y-k+1) - 2$ vertices. But G has clique number less than x and independence number at most $k-1+y-k = y-1$, so G must be an (x, y)-graph. This means that $R(x, y)$ must be greater than or equal to $R(x, k) + R(x, y-k+1) - 1$. ∎

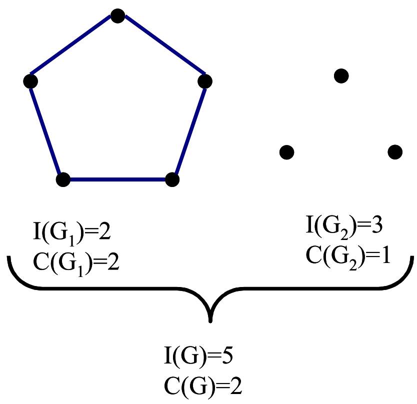

Figure 7. The disjoint union of an (x, k)-graph and an $(x, y-k+1)$-graph is an (x, y)-graph

4.3. **e-numbers and E-numbers.** It is often very useful to consider the number of possible edges in a graph, and for this purpose we will define some numbers.

The number $e(x, y, n)$ is defined as the minimum number of edges possible in an (x, y)-graph on n vertices. If there are no possible (x, y)-graphs on n vertices, $e(x, y, n)$ is defined to be ∞. This definition leads to the following relationship: For the lowest value of n for which $e(x, y, n) = \infty$, $R(x, y) = n$. A table of values for $e(x, y, n)$ is given as appendix B.

Sometimes it is also useful to calculate a similar number, E: $E(x, y, n)$ is the maximum number of edges possible in an (x, y)-graph on n vertices. A simple example to clarify this: for a graph on four vertices with no triangles and no independent set of size three or greater, we can see that there are exactly three possibilities:

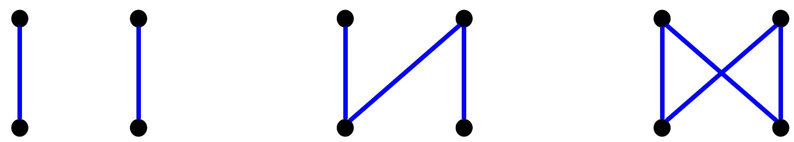

FIGURE 8. The three possible $(3, 3, 4)$-graphs.

From this we can conclude that $e(3, 3, 4) = 2$, and $E(3, 3, 4) = 4$.

4.4. **Upper and lower bounds for e and E.** As for Ramsey numbers there are several trivial values of e for small values of y and n:

- for $y > n$, $e(x, y, n)$ must be 0
- if $y = 1$ we would have to have a graph with no vertices, so $e = \infty$
- if $y = 2$ we cannot have any independent set of size 2, so all graphs must be complete if they exist at all:
  - if $n \leq x - 1$ we have a complete graph, so there are $\binom{n}{2}$ edges
  - if $n > x - 1$ we would have to have a graph on n vertices with no x-clique, which is impossible so $e = \infty$
- similarly for $x = 2$: if $n \leq y - 1$ we have no edges so $e = 0$, and for $n \geq y$ the e-number does not exist
- if $x = 3$ we have an upper bound for E: $\lfloor n(y - 1)/2 \rfloor$ since each vertex can have degree at most $y - 1$ (otherwise H_2 would be an independent y-set) and we can find the total maximum number of edges by observing that multiplying the n vertices by the maximum valence $y - 1$ counts each edge exactly twice.

Theorem 4. $e(x, y, n) \leq e(x, y - l + 1, n - m) + e(x, l, m)$

Proof. This is really the same theorem as theorem 2. If we examine the disjoint union of an $(x, y-l+1)$-graph on $n-m$ vertices and an(x, l, m)-graph on m vertices we can see that this must be an (x, y, n)-graph on n vertices. ∎

Theorem 5. *(Turán 1941)*

There is a graph $G = (V, E)$ with $|V| = n$ and $|E| = e$ which contains no k-clique iff

$0 \leq e \leq \binom{n}{2} - \frac{(q-1)n+(q+1)r}{2}$ *where $n = q(k - 1) + r, 0 \leq r < k - 1$. The upper equality holds iff $G = K_{q+1,\ldots,q+1,q,\ldots,q}$ with r copies of $q + 1$ and $k - 1 - r$ copies of q.*

Proof. The proof of this can be found in most books on graph theory. We will only give a proof for the case $k = 3$, since we will mostly be concerned with triangle-free graphs. In this case, the theorem reduces to $e \leq \lfloor \frac{n+1}{2} \rfloor * \lfloor \frac{n}{2} \rfloor$ ($e \leq \frac{n^2}{4}$ for even n, $e \leq \frac{n^2-1}{4}$ for odd n), and equality holds iff G is the complete bipartite graph $K_{\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$.

The proof is by induction on n: For $n = 3$ we have $e \leq 2$ and we can see that $K_{1,2}$ is a graph on 3 vertices with 2 edges and no triangles. For $n = 4$ we have that $e \leq 4$ and we can see that $K_{2,2}$ is the largest possible triangle-free graph on 4 vertices. Now let G be a graph on n vertices with e edges and no triangles, and assume the theorem for lower numbers of vertices.

Now, we have that each induced subgraph $G'$of G on $n-1$ vertices has at most $\lfloor \frac{n}{2} \rfloor * \lfloor \frac{n-1}{2} \rfloor$ edges by induction. We first assume that there is no subgraph with the maximum number of edges. We define the *density* of edges in a graph G as the number of edges in G divided by the number of edges in the complete graph on the same number of vertices, which can be seen as the probability of an edge existing between any two vertices. This gives us that the density of edges in G' must be $\leq \left(\lfloor \frac{n}{2} \rfloor * \lfloor \frac{n-1}{2} \rfloor - 1\right) / \binom{n-1}{2} = \left(\lfloor \frac{n}{2} \rfloor * \lfloor \frac{n-1}{2} \rfloor - 1\right) * \frac{2}{(n-1)(n-2)} \leq \frac{(n-1)^2-4}{4} * \frac{2}{(n-1)(n-2)} = \frac{(n^2-2n-3)}{2(n-1)(n-2)}$

which must be less than the total density of edges in G:

$e \leq \binom{n}{2} * \frac{(n^2-2n-3)}{2(n-1)(n-2)} = \frac{n(n-1)}{2} * \frac{(n^2-2n-3)}{2(n-1)(n-2)} = \frac{n(n^2-2n-3)}{4(n-2)} < \frac{n(n^2-2n)}{4(n-2)} = \frac{n^2}{4}$

For the second part of the theorem,. assume that there is a subgraph with the maximum number of edges. By induction it must then be $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor}$.

If we now add an nth vertex v to this graph , we see that v cannot have neighbors in both parts of the bipartite complete graph, since this would imply a triangle. Thus for the new graph to have the maximum possible number of edges, there must be an edge from v to every vertex in the larger part of the bipartite graph. But the graph so formed is exactly $K_{\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$. ∎

Theorem 6. *(This theorem is called the delta inequality and was first proved in [3].)*

If G is a $(3, y)$-graph on n vertices with e edges, then

$ne \geq \sum_{i=0}^{y-1} \left\{ e(3, y-1, n-i-1) + i^2 \right\} v_i$

where v_i is the number of vertices of degree i.

The non-negative function $ne - \sum_{i=0}^{y-1} \left\{ e(3, y-1, n-i-1) + i^2 \right\} v_i$ is often denoted by Δ.

Proof. First we prove a simple lemma: for any graph G,

$\sum_v \deg_2(v) = \sum_v (\deg(v))^2$

This is proved by counting the number walks of length 2 in G in two different ways. $\deg_2(v)$ is equal to the number of walks $v_1 - v_2 - v_3$ where $v_1 = v$, and $(\deg(v))^2$ is the number of walks $v_1 - v_2 - v_3$ where $v_2 = v$. Summing this over all vertices gives the desired result.

Now, if we have an (x, y)-graph we can prefer a vertex v as in section 4.1. The total number of edges in G must equal the number of edges in H_2 plus the number of edges that were removed, and this number is simply $\deg_2(v)$:

$\deg_2(v) + e(H_2) = e$
thus,
$\deg_2(v) + e(3, y-1, n{-}\deg(v) - 1) \leq e$

(If we prefer a vertex with degree $\deg(v)$, H_2 will have $n{-}\deg(v) - 1$ vertices since we have removed the vertex v and its neighbors, of which there are exactly $\deg(v)$.)

Summing this over all vertices, we get
$\sum\limits_v \deg_2(v) + e(3, y-1, n{-}\deg(v) - 1) \leq ne$
which, using our lemma, is equivalent to
$\sum\limits_v (\deg(v))^2 + e(3, y-1, n{-}\deg(v) - 1) \leq ne$

Now, if $\deg(v) = i$, and v_i is the number of vertices of degree i, we can sum the inequality over the possible valences for v: $\sum\limits_{i=0}^{y-1} \left\{ i^2 + e(3, y-1, n-i-1) \right\} v_i \leq ne$. ∎

Theorem 7. *(Radziszowski/Kreher 1988)*
For $k \geq 2$

$$e(3, k+1, n) = \begin{cases} 0 & \text{if } n \leq k \\ n-k & \text{if } k < n \leq 2k \\ 3n-5k & \text{if } 2k < n \leq 5k/2 \\ 5n-10k & \text{if } 5k/2 < n \leq 3k \end{cases}$$

when $n \geq 2.5k$ *we have that the minimum valence is* ≥ 2
for $n > 3k$, *we have that* $e(3, k+1, n) \geq 5n - 10k$.

We will first prove three lemmas. In each case, G is a minimum $(3, k{+}1, n)$-graph.

Lemma 3. *If a component of* G *is a cycle, then it is a pentagon.*

Proof. Let C_i be a cycle component in G of length i, $i \neq 5$. We know that $i \geq 4$ since G has no triangles. But if we now replace C_i with $i/2$ isolated edges if i is even, or a pentagon and $(i-5)/2$ isolated edges if i is odd, we will get a graph with fewer edges and the same independence number. ∎

Lemma 4. *If* G *has an isolated vertex, then all vertices in* G *are of degree less than 2.*

Proof. If v is an isolated vertex and w is a vertex of degree ≥ 2, we can delete all vertices adjacent to w and join v and w by an edge. Again, this is a $(3, k+1, n)$-graph with fewer edges than G. ∎

Lemma 5. *All vertices of degree exactly 1 are endpoints of isolated edges.*

Proof. If w is a vertex of degree ≥ 2 which is connected to a vertex v of degree 1, we can delete all other edges adjacent to w and we have a $(3, k+1, n)$-graph with fewer edges than G. ∎

Proof. Now we can return to the proof of the theorem. The proof is by classification of the possible graph structures. The case $n \leq k$ is obvious since a graph on n vertices with no edges still has no independent set of size $k+1$.

$k < n \leq 2k$:

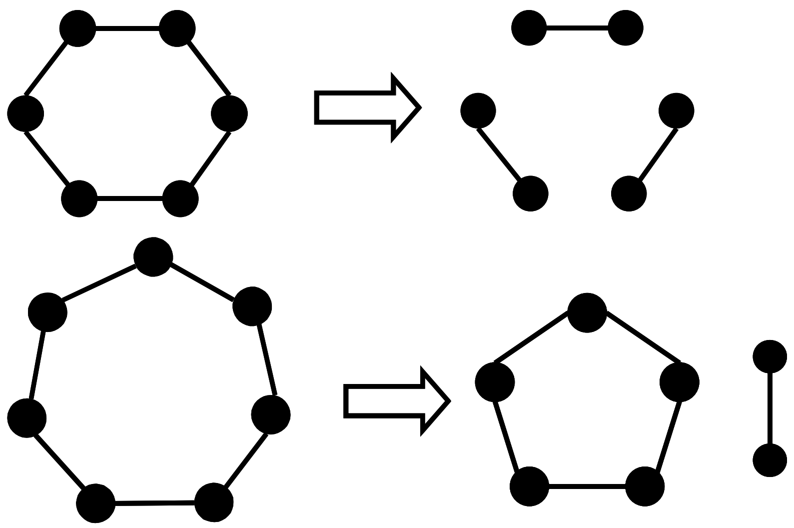

FIGURE 9. Graphs with the same independence number but fewer edges

We will prove that a minimum graph in this case must consist only of isolated vertices and isolated edges. Assume that the graph G is a minimum $(3, k+1, n)$-graph on $e \leq n-k$ edges, and let i be the number of isolated vertices in G. If $i > 0$, then by the second lemma above G consists only of isolated vertices and isolated edges. We then have that $i + e \leq k$ and $n = i + 2e$, which yields $e \geq n - k$. Thus we must have $e = n - k$ and $i = n - 2e = 2k - n$. Thus G must consist of $2k - n$ isolated vertices and $n - k$ isolated edges.

If $i = 0$, then G has only isolated edges and there are exactly $n - k$ of them.

$2k < n \leq 5k/2$:

For this case we use induction on k to prove that a minimum graph must consist only of pentagons and isolated edges. We begin by verifying the theorem for $k = 2$: Since $4 < n \leq 5$ we have $n = 5$ and thus $3n - 5k = 15 - 10 = 5$. This gives $e(3, 3, 5) = 5$ and we know that the pentagon is the unique $(3, 3)$ graph on 5 vertices. Now assume that G is a minimum $(3, k+1, n)$-graph with $e \leq 3n - 5k$ edges. There are two cases; either G has a vertex of degree 2 or it does not.

If there is no vertex of degree 2, then we must have at least one vertex v which has degree at least 3, since otherwise we would have only vertices of degree 0 or 1 and we would be in the case $n \leq 2k$. We have that $n-\deg(v) - 1 \leq n - 4 \leq 5k/2 - 4 \leq 5(k-1)/2$. If we prefer v in G and use the inductive assumption, H_2 is a $(3, k, n - deg(v) - 1)$-graph with $e(H_2) \geq 3(n-\deg(v) - 1) - 5(k-1)$ for $n-\deg(v) - 1 > 2(k-1)$ or $e(H_2) \geq n-\deg(v) - k$ for $n-\deg(v) - 1 \leq 2(k-1)$. In both these cases, $e(H_2) \geq e - 7$, which implies that $3 \leq \deg_2(v) \leq 7$. Thus v would have to have at least one vertex of degree 1 as a neighbor (since there is no vertex of degree 2) which contradicts part (c) of the lemma.

If there is a vertex v of degree 2, then if we prefer v H_2 is a $(3, k, n-3)$-graph with $e-\deg_2(v)$ edges. Now since $n \leq 5k/2$ we know that $n - 3 \leq 5(k-1)/2$ so by induction we have that the number of edges in H_2 is at least $3(n-3) - 5(k-1)$ for $n > 2k+1$ or at least $(n-3) - (k-1)$ for $n = 2k+1$. In both cases, since $e \leq 3n - 5k$, we have that $2 \leq \deg_2(v) \leq 4$. But if $\deg_2(v) \leq 3$ then v would have to be adjacent to at least one vertex of degree 1, which contradicts part (c) of our lemma since v has degree 2. Thus we must have $\deg_2(v) = 4$, and since this applies to all vertices of degree 2 (including v's neighbors) this implies that v is part of a cycle and we see that G must consist only of cycles and isolated edges. But lemma 1 implies that any cycle must be a pentagon, and thus by counting the total number of pentagons and isolated edges we can show that G must consist of exactly $5k-2n$ isolated edges and $n-2k$ pentagons. (Each isolated edge contributes 1 to a maximum independent set, while each pentagon contributes 2.) G is a $(3, k+1, n)$-graph with $e = 3n - 5k$ edges, since we in total have $2(5k - 2n) + 5(n - 2k) = n$ vertices, $5k - 2n + 5(n - 2k) = 3n - 5k$ edges, and the maximum independent set (given by the reasoning above) has $5k - 2n + 2(n - 2k) = k$ vertices.

$5k/2 < n$:

For the final two parts of this theorem, we will begin by proving the last part which states that $e(3, k+1, n) \geq 5n - 10k$ for $n \geq 2.5k$. (This statement is in fact true for all $n > 0$, which can be easily proved by applying the first three sections of the theorem.) After we have proved this we will construct a family of graphs for which the equality holds when $5k/2 < n \leq 3k$.

We use induction on k: For $k = 2$ we would have to have $n > 5$, but we know that there are no $(3, 3)$-graphs on more than 5 vertices. For $k = 3$ we have $n > 7.5$ and we know that $e(3, 4, 8) = 10$, (This is an immediate consequence of the trivial lower bounds and the delta inequality. A concrete proof can also be found in [3]) so the inequality holds for this case.

Now suppose G is a minimum $(3, k+1, n)$-graph for some $k \geq 4$, $n > 5k/2$ and $e < 5n - 10k$, and let v_i be the number of vertices in G of degree i. (Our goal is of course to prove that there can be no such graph) We now apply the delta inequality (theorem 6) and induction to G: $0 \leq \Delta(G, k+1, n, e) = ne - \sum_{i=0}^{k} v_i(i^2 + e(3, k, n - i - 1)) \leq ne - \sum_{i=0}^{k} v_i(i^2 - 5i + 5n - 10k + 5)$

Since $\sum_{i=0}^{k} v_i = n$ we have

$$0 \leq \Delta(G, k+1, n, e) \leq n(e - (5n - 10k - 1)) - \sum_{i=0}^{k} v_i(i-2)(i-3)$$

The coefficient $(i-2)(i-3)$ is always ≥ 0 (since i is an integer), so the above inequality and $e < 5n - 10k$ imply that $e = 5n - 10k - 1$. This gives us $\Delta(G, k+1, n, e) = 0$ which implies that G has only vertices of degree 2 and 3. Thus all the vertices of G are full (a full vertex is a vertex where the number of edges in $H_2(v) = e(x, y-1, n-\deg(v) - 1)$) and thus for any vertex v in G $\deg_2(v) = 4$ if $\deg(v) = 2$, or $\deg_2(v) = 9$ if $\deg(v) = 3$. Thus every component of G is either a cycle (if all vertices are of degree 2) or a 3-regular graph.

In the first case, we have by part (a) of the lemma that G is a disjoint union of a pentagon and some $(3, k-1, n-5, 5n-10k-6)$-graph, call it H. But by induction, H cannot exist since $5(n-5) - 10(k-2) > 5n - 10k - 6$.

The only possibility that remains is for G to be a 3-regular minimum $(3, k+1, n, e)$-graph, which has $e = 3n/2$ edges since it is 3-regular. But since we also have that $e = 5n - 10k - 1$, we must have $7n = 20k + 2$, or $k = 2 \bmod 7$. Thus we have that, for some $p \geq 0$, G is a 3-regular $(3, 10 + 7p, 26 + 20p, 39 + 30p)$-graph. To prove this part of our theorem, we must thus prove that no such graph can exist.

If we prefer a vertex v in G, we have that H_2 must be a $(3, k, \overline{n}, \overline{e})$-graph, where $\overline{n} = n - 4$ and $\overline{e} = 3n/2 - 9$ (since G is 3-regular). Note that H_2 is a minimum $(3, k, \overline{n})$-graph, since $e(3, k, n-4) \geq 5n - 10k - 10$ by induction, and since $k = (7n-2)/20$, this gives us $5n - 10k - 10 = 5n - (7n-2)/2 - 10 = 5n - 7n/2 - 9 = \overline{e}$.

We now show that H_2 can have only vertices of degree 2 and 3: obviously there are no vertices of degree greater than 3 since H_2 is a subgraph of the 3-regular graph G. If there were an isolated vertex, then by part (b) of the lemma H_2 would have at most $\overline{n}/2$ edges, which is a contradiction for $n \geq 26$ since we have $\overline{e} = 3n/2 - 9$. A vertex of degree 1 implies by part (c) of the lemma that H_2 is a union of an isolated edge and a $(3, k-1, \overline{n}-2, \overline{e}-1)$-graph. But such a graph cannot exist, since by induction we have that $e(3, k-1, \overline{n}-2) \geq 5n - 10k - 10 = \overline{e}$.

Thus H_2 has only vertices of degrees 2 and 3, and by counting edges we see that it has 6 vertices of degree 2 and $n - 10$ of degree 3. If we now examine a vertex u in H_2 which has degree 2, then we see that $\deg_2(u) \leq 5$ (with respect to H_2, not to G) since $e(3, k-1, \overline{n}-3) \geq 5(n-7) - 10(k-2) = \overline{e} - 5$. Thus any 2-vertex in H_2 has at least one neighbor of degree 2. If we consider the subgraph F induced by the six 2-valent vertices, we see that F cannot have isolated vertices since this would imply $\deg_2(u) = 6$. Nor can F have a pentagon, since there are six vertices altogether and this would imply an isolated vertex. But since all components of a graph with maximum valence 2 must be either cycles or paths, and we know from part (a) of the lemma that any cycle must be a pentagon, we see that the only possibility is for F to consist of a union of paths. We will now show that the only possibility here is for F to consist of exactly 3 isolated edges: If there exists a vertex w which is the starting vertex of a path in F of length greater than 2, we remove this vertex w and its neighbors, of which one is also in F. This then implies that the third vertex of this path, call it u, has degree at most 1 in the remainder of the graph H_2. But on the other hand, this remaining graph must have $\overline{\overline{n}} = n - 7$ vertices and independence number $\overline{\overline{k}} = k - 2$. We have removed 14 edges, which gives $\overline{\overline{e}} = e - 14 \leq 5n - 10k - 14 = 5(\overline{\overline{n}} + 7) - 10(\overline{\overline{k}} + 2) \leq 5\overline{\overline{n}} - 10\overline{\overline{k}}$. Thus $\overline{\overline{n}} - 2.5\overline{\overline{k}} = n - 7 - 2.5k + 5 \geq 1.5$ and since this is a minimal graph we have minimal valence at least 2 by induction.

Consequently, if any vertex v in G is preferred, the situation has to be as in figure 10.

It is clear from this figure that there are exactly 3 pentagons passing through each vertex v in G. Thus the total number of pentagons is $3n/5$ (since each pentagon is counted five times) which implies that n is a multiple of 5. But $7n = 20k + 2$ implies that $n \equiv 1 \bmod 5$. Thus the graph G cannot exist, and we have proved the inequality.

To prove equality in the case $5k/2 < n < 3k$, we must construct $(3, k+1, n, 5n - 10k)$-graphs. For low values of k the constructions are as follows:

$k = 2$: The only possibility is a $(3, 3, 5, 5)$-graph, which of course is a pentagon.

$k = 3$: The only possibility here is a $(3, 4, 8, 10)$-graph.

$k = 4$: The two possible parameter situations are $(3, 5, 11, 15)$ and $(3, 5, 12, 20)$.

$k = 5$: The possibilities are $(3, 6, 13, 15)$ and $(3, 6, 14, 20)$.

The graphs for $k = 3, 4$ and 5 are shown in figure 11.

For $k \geq 6$ we consider the disjoint union of a pentagon and any minimum $(3, k-1, n-5)$-graph H. By induction H has $5n - 10k - 5$ edges since $5(k-2)/2 < n - 5 < 3(k-2)$, so the graph thus obtained is a $(3, k+1, n, 5n - 10k)$-graph.

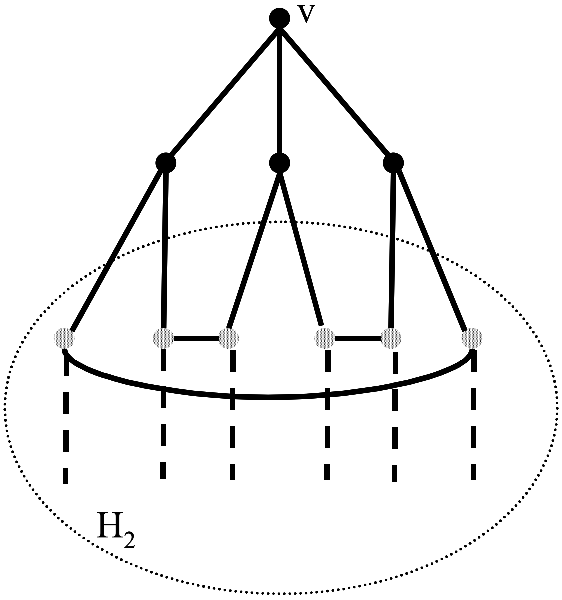

FIGURE 10. The graph G: The highlighted vertices are the subgraph F

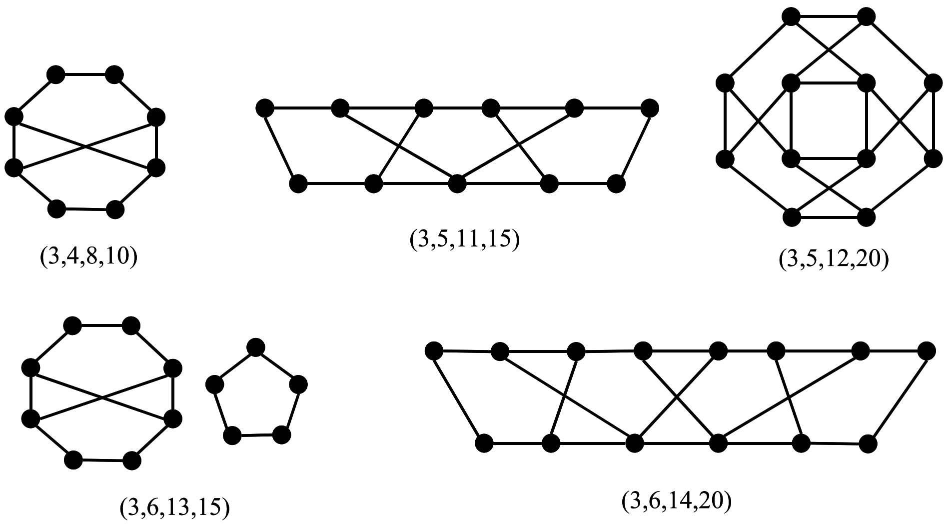

FIGURE 11. $(3, k+1, n, 5n-10k)$-graphs for $k = 3, 4$ and 5

For the case $n = 3k$ we introduce a family of graphs known as *bicycles*. (The name was coined by Backelin). The bicycle $BC_m, m \geq 4$, is defined as the graph on the $3m$ vertices $u_1, ..., u_m, v_1, ..., v_m, ..., w_1, ..., w_m$ and with the $5m$ edges $\{u_i, v_i\}$,

$\{u_i, w_i\} \forall i$ and $\{w_i, w_j\}, \{w_i, v_j\}, \{v_i, u_j\}$ for $j - i \equiv 1 \bmod m$. BC_m has $3m$ vertices, $5m$ edges, and no triangles, so to prove that BC_m is a $(3, m+1, 3m, 5m)$-graph we must show that it has no independent set of size $m + 1$. We do this by examining the 4-cycles $\{w_k, w_{k+1}, v_k, u_k\}$, call them C_k. Any independent set I can contain at most two vertices from any C_k. But if w_k and v_k are both in I, then C_{k-1} cannot contain any other independent vertices than w_k. Denote the number of independent vertices contributed to I in this way by α.Similarly, if w_{k+1} and u_k are both in I, then there can be no other contribution from C_{k+1}. Denote the number of vertices contributed in this way by β. All of the other C_k's can at most contribute one vertex to I, and there are $k - \alpha - \beta$ of them. Thus, $|I| \leq \alpha + \beta + k - \alpha - \beta = k$, and thus there can be no independent set of size $k + 1$. BC_m is a minimum graph since $e = 5m = 5(3m) - 10m = 5n - 10k$ which is exactly the lower bound given by our theorem.

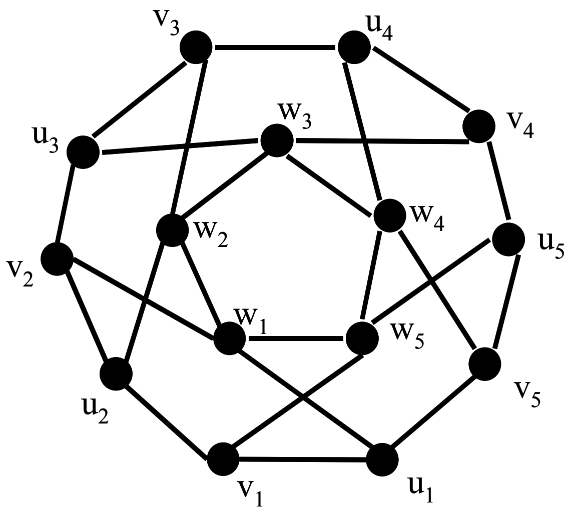

FIGURE 12. The bicycle BC_5

∎

Theorem 8. *(Radzisowski/Kreher 1991)*
$e(3, k+1, l) \geq 6n - 13k$ *for all* $k, n > 0$

This theorem was proved by Radziszowski and Kreher in [9], using similar methods as for the above theorem. It is included in FRANK for completeness and influences higher estimates of bounds for e-numbers, but is not necessary for the work done in the following section.

Theorem 9. *(Backelin)*
$e(3, k+1, l) \geq (40n - 91k)/6$ *for all* $k, n > 0$

This was proved by Backelin in [1]. It improves some e-number bounds for $k \geq 11$ and is also included in FRANK.

5. A new bound for R(3,12)

In this chapter we improve the bound $R(3,12) \leq 60$ to $R(3,12) \leq 59$, mainly by proving that the bound $(3,11,48) \geq 214$ can be sharpened to 215. All values and bounds used in this chapter are those calculated by FRANK, using as input values from [3], [4], [5], [6], [7] and [8]. Many of the techniques used here could of course be used to improve other similar bounds.

5.1. **Basic techniques.** The delta inequality, theorem 6, can be used to calculate bounds on the degrees of vertices in an (x,y)-graph as follows:

We begin by calculating the *central valence configuration*: The valence configuration that gives us the most even distribution of valences possible, with at most two different valences (which must be consecutive).

We know that any valence configuration must satisfy the following two equations (v_i is the number of vertices of degree i):

$$\begin{cases} \sum v_i = n \\ \sum i v_i = 2e \end{cases}$$

(The first equation simply sums all the vertices, and the second comes from the fact that if we count the total number of edges incident to each vertex, we will have counted each edge twice.)

For the central valence configuration, we have at most two consecutive valences and this becomes

$$\begin{cases} v_i + v_{i+1} = n \\ i v_i + (i+1) v_{i+1} = 2e \end{cases}$$

which has the solution

$i = \left\lfloor \frac{2e}{n} \right\rfloor, i+1 = \left\lfloor \frac{2e}{n} \right\rfloor + 1, v_i = n - v_{i+1}, v_{i+1} =rem(2e/n)$

We represent vertex configurations as vectors, where the number in position i represents v_i. Note that, since it is possible for a vertex to have valence 0, these vectors have a zeroth position. (A simple example: a pentagon has the valence configuration vector $(0,0,5)$)

Now, if we have a vertex v with valence i in an (x,y)-graph G, we can calculate the theoretical maximum value for the second valence $\deg_2(v)$. We note that this theoretical bound is given by $e(G) - e(H_2(v))$, since if we prefer any vertex in an (x,y)-graph, $H_2(v)$ has exactly $e(G) - \deg_2(v)$ edges. In practice we of course often do not know the exact value of $e(H_2(v))$, but we can substitute a lower bound, call it $elb(H_2(v))$, since we are interested in the maximum possible value of $\deg_2(v)$. But we also know that for any vertex v in any graph, $\sum_v \deg_2(v) = \sum_v (\deg(v))^2$. If we now replace $\deg_2(v)$ with our theoretical maximum bound, we have the following:

$\sum_v (e(G) - elb(H_2(v)) - (\deg(v))^2) \geq 0$. We will call this the *delta surplus* for G, since if we have exact values this is the delta inequality, theorem 6. This delta surplus is convenient for keeping track of the possible valence configurations in an (x,y,n)-graph, since it provides an upper bound for the number of possible vertices of different degrees. We must first show that the central valence configuration discussed above is the configuration which gives the maximum delta surplus, and for this we will need the following definition:

Definition 1. *The sequence $a_1, ..., a_m$ is sufficiently convex if the following condition is satisfied: $a_i + a_{i-2} \geq 2a_{i-1} - 2$ for $i = 3, ..., m$.*

If we now have a sequence of lower bounds, which depend only on the valence of v, for the various values of $e(H_2(v))$ in an (x, y, n)-graph which is sufficiently convex, this implies that adding $\deg(v)^2$ to each element in this sequence will produce a sequence which is convex. Now this implies that the sequence $e(G) - e(H_2(v)) - (\deg(v))^2$ is concave, so the maximum delta surplus is given by the central valence configuration.

5.2. **Investigation of** $R(3, 12)$**.** From our tables of results we can see that if $e(3, 12, 59) \neq \infty$, then $322 \leq e(3, 12, 59) \leq 324$. Since this is a relatively small interval it seems feasible to investigate all the possible graphs by hand. For each possible configuration, we prefer a vertex v and examine the number of edges in H_2, which we denote by $e(H_2)$. Since we know that H_2 must be a $(3, 11)$-graph on $59 - \deg(v) - 1$ vertices (we have removed the vertex v and its $\deg(v)$ neighbors), we can compare $e(H_2)$ with our tabulated values for $e(3, 11, 59 - \deg(v) - 1)$ to see which configurations are possible.

For $e = 324$ there is only one possible valence configuration: $(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 58)$. Preferring the 10-valent vertex gives $e(H_2) \leq 324 - (10 * 11) = 214$.

For $e = 323$ there are three possibilities:

$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 58)$:

If we prefer the 8-valent vertex then $e(H_2) \leq 323 - (8*11) = 235$ but $e(3, 11, 50) \geq 242$, so this configuration cannot be possible.

$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 57)$:

If we prefer the 9-valent vertex then $e(H_2) \leq 323 - (9 + 8 * 11) = 226$ but $e(3, 11, 49) \geq 228$.

$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 56)$:

The three 10-valent vertices cannot form a triangle, so at least one of them must have no more than one other 10-valent neighbor. If we prefer this vertex then $e(H_2) \leq 323 - (10 + 9 * 11) = 214$.

For $e = 322$ there are exactly seven possible valence configurations:

$(0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 58)$:

If we prefer the 6-valent vertex then H_2 would be a $(3, 11, 52)$-graph, which we know is impossible ($e(3, 11, 52) = \infty$).

$(0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 57)$:

Preferring the 7-valent vertex gives $e(H_2) \leq 322 - (10 + 6 * 11) = 246$ but $e(3, 11, 51) \geq 255$.

$(0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 57)$

Preferring the 8-valent vertex gives $e(H_2) \leq 322 - (9 + 7 * 11) = 236$ but $e(3, 11, 50) \geq 242$.

$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 2, 56)$

Preferring the 8-valent vertex gives $e(H_2) \leq 322 - (2 * 10 + 6 * 11) = 236$ but $e(3, 11, 50) \geq 242$.

$(0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 1, 56)$:

Preferring a 9-valent vertex gives $e(H_2) \leq 322 - (9 + 10 + 7 * 11) = 226$ but $e(3, 11, 49) \geq 228$.

$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 55)$:

Preferring the 9-valent vertex gives $e(H_2) \leq 322 - (3 * 10 + 6 * 11) = 226$ but $e(3, 11, 49) \geq 228$.

$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 54)$:

We observe that G_{10}, the induced subgraph consisting of the 10-valent vertices, must contain at least one vertex with degree 2 or less by Turàn's theorem. Preferring this vertex gives $e(H_2) \leq 322 - (2*10 + 8*11) = 214$.

In other words, if there is a $(3,12)$-graph on 59 vertices then $H_2(v)$ for any vertex v in this graph must be a $(3,11,48,214)$-graph, so if we can prove that $e(3,11,48) > 214$, then $e(3,12,59)$ must be ∞, which implies that $R(3,12)$ must be ≤ 59.

5.3. **Improving the bound for** $e(3,11,48)$**.** In this section we will prove the nonexistence of a $(3,11,48,214)$-graph by similar methods as used above. To keep track of the configurations we use a technique called *delta surplus investigation.*

- The central valence configuration given by the equations in section 5.1 is $i = \lfloor 2*214/48 \rfloor = 8, i+1 = 9, v_i = 4, v_{i+1} = 44$, or $(0,0,0,0,0,0,0,0,4,44,0)$.
  This configuration gives a delta surplus of $\sum_v (e(G) - e(H_2(v)) - (\deg(v))^2) = 4*((214-145)-8^2) + 44((214-133)-9^2) = 20$.
- Now, we examine the *cost* for vertices of different degrees. We create a table which corresponds to the bracketed expression on the right-hand side of the delta inequality. We start with the best known estimates of e(3,10,48-i-1) (the maximum degree for any vertex is of course 10 since a vertex with degree 11 in a triangle-free graph would give us an independent 11-set):
  $(\infty,\infty,\infty,\infty,\infty,182,169,156,145,133,123)$
  Now we can add i^2 to each estimate, which ensures that we have convexity since the finite part of the degree sequence is sufficiently convex.
  $(\infty,\infty,\infty,\infty,\infty,182+5^2,169+6^2,156+7^2,145+8^2,133+9^2,123+10^2)$
  $= (\infty,\infty,\infty,\infty,\infty,207,205,205,209,214,223)$
  Now we can calculate the *cost* for a vertex of degree v: For example if we have a vertex of degree 7, the simplest possible configuration would be $(0,0,0,0,0,0,0,1,2,45,0)$. The delta surplus in this case is $1*((214-156)-49) + 2*(214-145)-64) + 45*(214-133)-81) = 19$. Since we had a maximum delta surplus of 20, we can express this as "the cost for a vertex of degree 7 is 1". Similar calculations for vertices of other degrees give us the following "cost table":

Degree:	≤ 4	5	6	7	8	9	10
Cost:	∞	13	6	1	0	0	4

- From the above table we can immediately draw conclusions about the possible vertex configurations, for example we can see that there can be at most one vertex of degree 5, since otherwise we would have a cost that is higher than the maximum possible surplus of 20.
- The next step is to examine all the possible vertex combinations to see if they are possible (3,11)-graphs. The goal is of course to eliminate all the possible configurations, which leads to the conclusion that there are no possible$(3,11,48)$-graphs with 214 edges, and thus $e(3,11,48) \geq 215$. Our main method is the same as in the previous section: we examine the number of edges in H_2 when a vertex v is preferred, and show that this leads to a contradiction in each case. We have that $\deg_2(v) + e(H_2) = e$ and thus $\deg_2(v) = e - e(H_2)$. Thus, if $\deg_2(v)$ is greater than 214 minus the bound for $e(H_2)$ given in our tables then that particular configuration is impossible.

- We begin by examining the configurations which contain a vertex of degree 5. For such a configuration to be possible we must have $\deg_2(v) \leq 214 - e(3,10,42) = 214 - 182 = 32$. We know from the cost calculations above that there are only two possible cases when we have a vertex of degree 5: either we have one vertex of degree 6, and in that case no more than one vertex of degree 7, or no vertices of degree 6 and at most 7 of degree 7. In the first case, preferring the vertex of degree 5 would give us $\deg_2(v) \geq 6 + 7 + 3 * 8 = 37$. The second case gives us $\deg_2(v) \geq 5 * 7 = 35$. Thus, there are no possible $(3, 11, 48)$-graphs on 214 edges that contain a vertex of degree 5.
- We now continue with vertices of degree 6. From the table above, we can see that there can be at most three vertices of degree 6. To keep track of the separate cases we use the notation v_i for the number of vertices of degree i. If we have exactly three vertices of degree 6, the equations discussed in section 5.1 together with the fact that we can have at most an additional cost of 2 (since the vertices of degree 6 give a total cost of 18) give us the following system:
$$\begin{cases} 7v_7 + 8v_8 + 9v_9 + 10v_{10} = 428 - 3 * 6 \\ v_7 + v_8 + v_9 + v_{10} = 48 - 3 \\ v_7 + 4v_{10} \leq 20 - 3 * 6 \end{cases}$$
From this we can see immediately that $v_{10} = 0$, so we have no vertices of degree 10. The rest of the system reduces to
$$\begin{cases} 2v_7 + v_8 = -5 \\ v_9 = 45 - v_7 - v_8 \end{cases}$$
which has no non-negative solutions. Thus we have eliminated the possibility of three 6-valent vertices. Two 6-valent vertices give the system
$$\begin{cases} 7v_7 + 8v_8 + 9v_9 + 10v_{10} = 428 - 2 * 6 \\ v_7 + v_8 + v_9 + v_{10} = 48 - 2 \\ v_7 + 4v_{10} \leq 20 - 2 * 6 \end{cases}$$
and we can see that v_{10} must be either 0, 1 or 2. For $v_{10} = 0$ there are no solutions, while $v_{10} = 1$ and $v_{10} = 2$ give the possible configurations $(0, 0, 0, 0, 0, 0, 2, 0, 1, 44, 1)$ and $(0, 0, 0, 0, 0, 0, 2, 0, 0, 44, 2)$ respectively. In each of these, we can prefer a vertex of degree 6 and calculate $\deg_2(v)$ as above. Since $e(3, 10, 41) \geq 169$ we know that for a configuration to be possible we must have $\deg_2(v) \leq 214 - 169 = 45$. For the first configuration above, $\deg_2(v) \geq 6 + 8 + 4 * 9 = 50$ (we have removed as few edges as possible). The second case gives us $\deg_2(v) \geq 6 + 5 * 9 = 51$ so this is not a possibility either.
Exactly one 6-valent vertex would give the system
$$\begin{cases} 7v_7 + 8v_8 + 9v_9 + 10v_{10} = 428 - 6 \\ v_7 + v_8 + v_9 + v_{10} = 48 - 1 \\ v_7 + 4v_{10} \leq 20 - 6 \end{cases}$$
which after some calculation gives us the possible configurations
$(0, 0, 0, 0, 0, 0, 1, 0, 1, 46, 0)$, $(0, 0, 0, 0, 0, 0, 1, 0, 2, 44, 1)$, $(0, 0, 0, 0, 0, 0, 1, 1, 0, 45, 1)$, $(0, 0, 0, 0, 0, 0, 1, 0, 3, 42, 2)$, $(0, 0, 0, 0, 0, 0, 1, 1, 1, 43, 2)$, $(0, 0, 0, 0, 0, 0, 1, 0, 4, 40, 3)$, $(0, 0, 0, 0, 0, 0, 1, 1, 2, 41, 3)$, and $(0, 0, 0, 0, 0, 0, 1, 2, 0, 42, 3)$
Preferring the 6-valent vertex, we can see that the smallest possible value for $\deg_2(v)$ is
$7+2*8+3*9 = 50$ (this is given by both the sixth and seventh configuration in the above list) and since $\deg_2(v)$ had to be less than 45 we see that there are no possible configurations with exactly one 6-valent vertex.

- We now examine configurations with only vertices of degree 7 or higher. If we prefer a vertex of degree 7, H_2 must be a $(3, 10, 41)$-graph and thus have at least $e(3, 10, 41) \geq 156$ edges, and so $\deg_2(v) \leq 214 - 156 = 58$.

$$\begin{cases} 7v_7 + 8v_8 + 9v_9 + 10v_{10} = 428 \\ v_7 + v_8 + v_9 + v_{10} = 48 \\ v_7 + 4v_{10} \leq 20 \end{cases}$$

If we have at least one vertex with degree 7, this means that we can have at most four with degree 10. Then there are four possible configurations: $(0, 0, 0, 0, 0, 0, 0, 4, 0, 40, 4)$, $(0, 0, 0, 0, 0, 0, 0, 3, 2, 39, 4)$, $(0, 0, 0, 0, 0, 0, 0, 2, 4, 38, 4)$ and $(0, 0, 0, 0, 0, 0, 0, 1, 6, 37, 4)$.

$(0, 0, 0, 0, 0, 0, 0, 4, 0, 40, 4)$: It would seem at first that $\deg_2(v) \geq 3*7+4*9 = 57$ which would mean that this configuration is possible. However, we can observe that it is not possible for all of the vertices of degree 7 to have three other 7-valent vertices as neighbors, since then these vertices would form a 4-clique. Thus we can be certain that at least one of the 7-valent vertices has no more than two 7-valent neighbors, which gives us $\deg_2(v) \geq 2*7+5*9 = 59$ and we have eliminated this configuration.

$(0, 0, 0, 0, 0, 0, 0, 3, 2, 39, 4)$: The 7-valent vertices cannot form a triangle, and so at least one of them has no more than one 7-valent neighbor. Preferring this vertex gives us $\deg_2(v) \geq 7 + 2*8 + 4*9 = 59$

$(0, 0, 0, 0, 0, 0, 0, 2, 4, 38, 4)$: For $\deg_2(v)$ to be less than or equal to 58, a 7-valent vertex must have the other 7-valent vertex and at least three of the four 8-valent vertices as neighbors. However, if we have exactly this situation then we can prefer the other 7-valent vertex, which gives us $\deg_2(v) \geq 7+8+5*9 = 60$. (See figure 13)

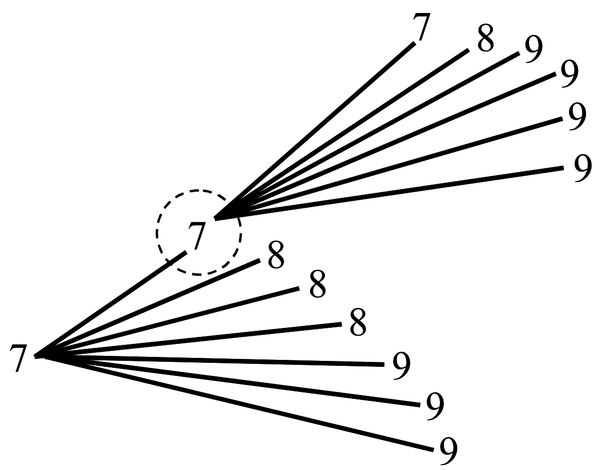

FIGURE 13. The first vertex of degree 7 has $\deg_2(v) \leq 58$, but the circled vertex must then have $\deg_2(v)$ at least 60.

- $(0, 0, 0, 0, 0, 0, 0, 1, 6, 37, 4)$: There are several possibilities here. If the 7-valent vertex has all of the 8-valent vertices as neighbors, then $\deg_2(v) = 6*8+9 = 57$. To see that this is not a possibility we can instead prefer one of the 8-valent vertices, which then has $\deg_2(v) \geq 7 + 7 * 9 = 70 > 214 - e(3, 10, 39) = 69$. If the 7-valent vertex instead has only five 8-valent neighbors, then we have two possible cases: Either one of these vertices has no other 8-valent neighbor, in which case we can prefer this vertex and $\deg_2(v) \geq 70$ as above, or all of the five have the same 8-valent neighbor (since there are only six totally). But preferring the 7-valent vertex here would remove all of the gray edges in the figure, and this would leave us with a vertex with valence ≤ 3 in H_2. But H_2 is a (3,10,39,145)-realizer, and it is possible to prove that such a graph can have no vertices of degree less than 6. This proof is given in section 5.4.

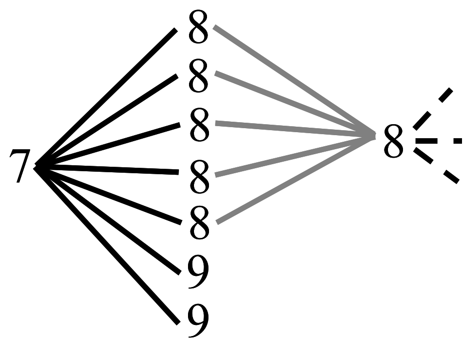

FIGURE 14. The 8-valent vertex on the right has degree less than 6 in H_2

Three 10-valent vertices give the possible configurations $(0, 0, 0, 0, 0, 0, 0, 1, 5, 39, 3)$, $(0, 0, 0, 0, 0, 0, 0, 2, 3, 40, 3)$ and $(0, 0, 0, 0, 0, 0, 0, 3, 1, 41, 3)$. The last one can be eliminated by taking into account that the three 7-valent vertices cannot form a triangle, so there must be at least on of them that has no more than one 7-valent neighbor. Preferring this vertex gives us $\deg_2(v) \geq 7 + 8 + 5 * 9 = 60$. The first two require slightly more work: in the first case, we note that for $\deg_2(v) \leq 58$ for the vertex of degree 7, it must have all of the 8-valent vertices as neighbors. In this case we can prefer one of the 8-valent vertices and eliminate this case by the same argument used previously: $\deg_2(v) \geq 7 + 7 * 9 = 70$. In the second case, a 7-valent vertex with $\deg_2(v) \leq 58$ would have the other 7-valent vertex and all of the three 8-valent vertices as neighbors, and exactly the same argument can be applied.

Two 10-valent vertices give the possibilities $(0, 0, 0, 0, 0, 0, 0, 1, 4, 41, 2)$, $(0, 0, 0, 0, 0, 0, 0, 2, 2, 42, 2)$ and $(0, 0, 0, 0, 0, 0, 0, 3, 0, 43, 2)$. These can easily be eliminated; $\deg_2(v) \geq 4*8+3*9 = 59$, $7+2*8+4*9 = 59$ and $7+6*9 = 61$ respectively.

One 10-valent vertex gives us two possible configurations: $(0, 0, 0, 0, 0, 0, 0, 1, 3, 41, 1)$ and $(0, 0, 0, 0, 0, 0, 0, 2, 1, 44, 1)$. Each of these gives $\deg_2(v) \geq 60$ so we can eliminate these.

There are also two possible configurations with no 10-valent vertices: $(0, 0, 0, 0, 0, 0, 0, 1, 2, 45)$ and $(0, 0, 0, 0, 0, 0, 0, 2, 0, 46)$, but these each give $\deg_2(v) \geq 61$.

- There is only one possible configuration with only vertices of degrees exactly 8 or 9: $(0,0,0,0,0,0,0,0,4,44)$. For this to be possible $\deg_2(v)$ would have to be $\leq 214 - e(3,10,39) = 69$. We know that at least one of the four 8-valent vertices has two or less 8-valent neighbors, since otherwise we would have a triangle, and if we prefer this vertex we get $\deg_2(v) \geq 2*8+6*9 = 70$, so this is not a possibility.
- We now examine configurations with vertices of degree 10 (and no vertices of lower degree than 8 since these have already been eliminated). One 10-valent vertex gives us $(0,0,0,0,0,0,0,0,0,5,42,1)$ as the only possible configuration. Preferring an 8-valent vertex, we see that $\deg_2(v) \leq 69$.By the same argument we have used above, there must be at least one vertex among the five 8-valent vertices that has no more than two 8-valent neighbors (the induced subgraph formed by the 8-valent vertices can at most be $K_{2,3}$). Preferring this vertex gives $\deg_2(v) \geq 2*8+6*9 = 70$.

  Two 10-valent vertices give the configuration $(0,0,0,0,0,0,0,0,6,40,2)$. If we examine G_8, the induced subgraph on the 8-valent vertices, we can see that the only possibility that does not immediately give us $\deg_2(v) > 69$ is if $G_8 = K_{3,3}$. But again H_2 is a $(3,10,39,145)$-realizer and thus has no vertices of degree less than 6. (section 5.4) If we prefer any of the six vertices of degree 8 we have the situation shown in figure 15, and we can see that removing these edges would leave us with two vertices in H_2 which each have degree less than 6, and so we have eliminated this configuration.

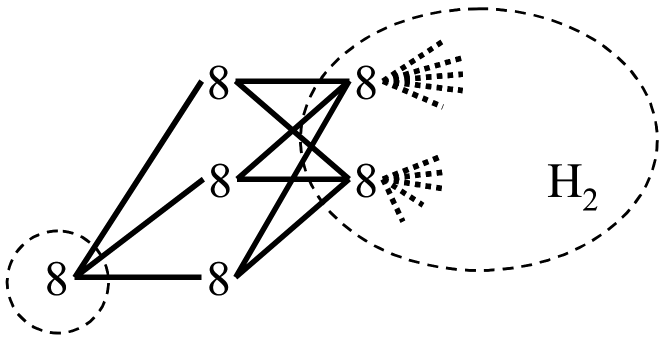

FIGURE 15. If $G_8 = K_{2,3}$ we have vertices in H_2 with degree less than 6

Three 10-valent vertices give us the configuration $(0,0,0,0,0,0,0,0,7,38,3)$. There are two cases where we do not have $\deg_2(v) > 69$: either G_8 has 11 or 12 edges. To eliminate these possibilities by a similar argument as in the previous case we need to show that there exists a vertex in G_8 with $\deg_2(v) = 69$, the removal of which would imply a vertex in H_2 (which is a $(3,10,39,145)$-graph) with degree less than 6. In other words, we must show that G_8 contains $K_{2,3}$ as a subgraph. But if G_8 is a graph on 7 vertices with 12 edges (and no triangles) , then by Turàn's theorem it must be isomorphic to $K_{3,4}$, which obviously contains $K_{2,3}$ as a subgraph. If we have 11 edges slightly more work is required: Assume that there is a graph G on 7 vertices with 11 edges that contains no triangles and does not have $K_{2,3}$ as a subgraph. We now examine any induced subgraph H on 5 of the 7 vertices in G: by Turàn's theorem H can have at most 6 edges, and is then isomorphic to $K_{2,3}$, so if we are to avoid $K_{2,3}$ this means that there can be at most 5 edges in H. Now, we can think of the graph G as being the blue edges in a blue-red coloring of K_7. Now each subgraph H, which is a blue-red K_5, can contain at most 5 blue edges out of the 10 possible edges in K_5. So in the entire graph G, the number of blue edges can at most be equal to the number of red edges. But this implies that there are at most $\lfloor \frac{21}{2} \rfloor = 10$ blue edges in total, and we have a contradiction. Thus, if the graph G_8 is triangle-free it must contain $K_{2,3}$ as a subgraph (and we can also see that since G_8 has 11 edges there must be 6 vertices with degree 3 in G_8, and so at least one of these must be part of the subgraph $K_{2,3}$) and we can prefer one of these three-valent vertices and are left with a vertex in H_2 with degree less than 6.

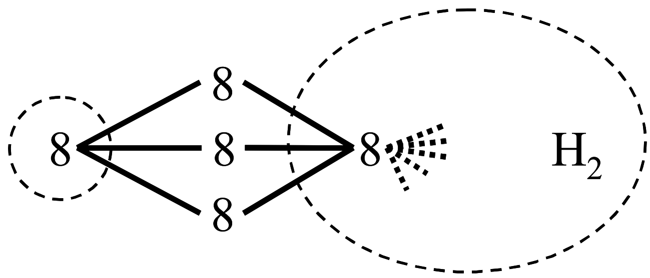

FIGURE 16. If G_8 contatins $K_{2,3}$ as a subgraph, preferring one of the two vertices of degree 3 leaves a vertex with degree less than 6 in H_2.

- For the two remaining configurations, $(0,0,0,0,0,0,0,0,8,36,4)$ and $(0,0,0,0,0,0,0,0,9,34,5)$, we need to introduce a new trick.

  We define the *extra cost* for a vertex as the difference between the maximum possible valence and the actual valence, or

  $\textit{extra cost} = e(H_2) - e(3, y-1, n{-}\deg(v) - 1)$

  The total cost must be equal to the delta surplus, so in our situation we know that for any configuration the sum of the basic costs and the extra costs is 20.

  So, if we have a vertex of valence 9 in our case, the extra cost is $e(H_2) - e(3, 10, 48-9-1) = e(H_2) - 133$

  which means that for each extra edge in H_2 we have an extra cost of 1. For $e(H_2)$ to be exactly 133,

  $\deg_2(v) = 81$. We now examine the possible neighbors for a 9-valent vertex: If we have one or more 8-valent vertices and no 10-valent vertices as neighbors, then $\deg_2(v) < 81$ which would give an extra cost. If we allow both 8- and 10-valent vertices as neighbors we have the following picture:We see that the circled vertices cannot form an independent set since there are 11 of them in total, and thus there must be at least one edge between them, which is dashed in the figure. (Since there can be no triangles, an edge between two of the neighbors of the 9-valent vertex is not a possibility). In this situation it could be that $\deg_2(v) = 81$, but we note that if we prefer the 9-valent vertex then it's 8-valent neighbor would have degree 6 or less in H_2. However, this is impossible since a $(3, 10, 38, 133)$-graph must be 7-regular (the central valence configuration is $(0, 0, 0, 0, 0, 0, 0, 38)$ and we have equality in the delta inequality, so since there is no delta surplus this must be the only possible configuration). Thus all 9-valent vertices must have only 9-valent neighbors or an extra cost of at least 1 if it has one or more 8-valent neighbors.

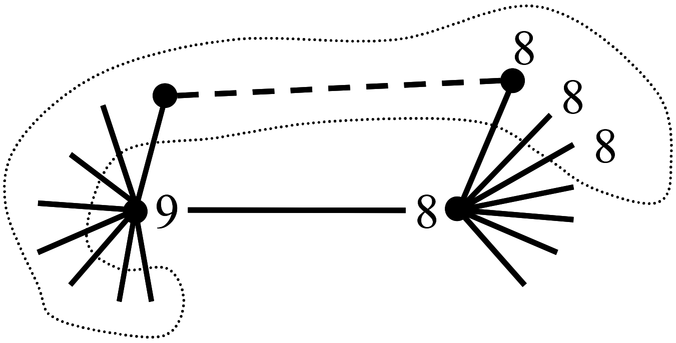

FIGURE 17. A 9-valent vertex with neighbors of other valencies must have an extra cost

We now examine the two configurations.

$(0, 0, 0, 0, 0, 0, 0, 0, 9, 34, 5)$:

This configuration has a basic cost of exactly 20, since each of the five 10-valent vertices contributes 4. This means that we can have no extra costs at all. If we now examine G_8 we see that each vertex must have $\deg_2(v) \leq 214 - e(3, 10, 39) = 69$, and since we have no 9-valent neighbors we can see that the maximum number of 10-valent neighbors for any vertex in G_8 is two. Thus the minimum valence for G_8 is ≥ 6, which means that G_8 has at least $\frac{6*9}{2} = 27$ edges. But by Turán's theorem we know that the maximum number of edges in a triangle-free graph on 9 vertices is $\binom{9}{2} - \frac{3*9+5*1}{2} = 20$. Thus this configuration is impossible.

$(0, 0, 0, 0, 0, 0, 0, 0, 8, 36, 4)$:

Since the basic cost for this configuration is 16, we can have extra costs of at most 4, and so we must prove that there are at least five 9-valent vertices which have 8-valent neighbors. As above, we know that if we have an 8-valent vertex with $\deg_2(v) > 69$ we can prefer this vertex and thus eliminate this configuration, so the only possibility is for all 8-valent vertices to have $\deg_2(v) \leq 69$. If these vertices have no 9-valent neighbors, then they can have at most two 10-valent neighbors and hence at least six 8-valent ones. But by the same argument as above, this would mean that G_8 has at least $\frac{6*8}{2} = 24$ edges, but Turán's theorem gives $\binom{8}{2} - \frac{3*8+5*0}{2} = 16$. With at most two 10-valent neighbors, the only possibility for the second degree of the 8-valent vertex is $4 * 8 + 10 + 3 * 9 = 69$. However, this gives an extra cost of 3. In this same situation, we examine one of this vertex's 8-valent neighbors. It can have no more than three other neighbors of degree 8 (since there are eight 8-valent vertices), and at most one neighbor of degree 10 since $\deg_2(v) \leq 69$. However, this leaves at least three vertices of degree 9 (which are different from the previous 9's since otherwise we would have a triangle) and thus an additional extra cost of three. Hence we have an extra cost of 6, which is impossible.

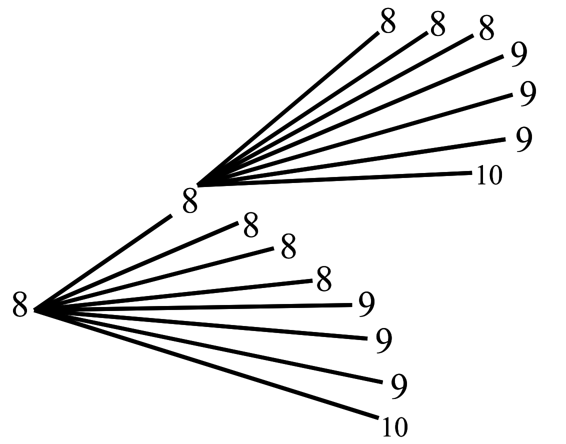

FIGURE 18. A subgraph which gives an extra cost of 6

- Thus, we have proved that there are no possible vertex configurations which give us a $(3, 11, 48)$-graph with 214 edges, and so $e(3, 11, 48) \geq 215$.

5.4. **A proof that a $(3, 10, 39, 145)$-realizer has no vertices of degree less than 6.** By the same methods as above, we can calculate the costs for vertices of different degrees. The total delta surplus is 15, and the cost table is as follows:

Degree:	4	5	6	7	8	9
Cost:	17	6	3	0	0	2

From this we can see that there can be at most two vertices of degree 5. Exactly two would mean that we can have at most one vertex of degree 6, and preferring one of the 5-valent vertices gives us $e(H_2) \leq 145 - (5 + 6 + 3 * 7) = 113$, which is a contradiction since we know that $e(3, 9, 33) \geq 114$.

Exactly one 5-valent vertex would mean that we can have no more than three of degree 6, and we have $e(H_2) \leq 145 - (3 * 6 + 2 * 7) = 113$, again a contradiction.

6. Computational Methods

All the computations in this paper have been carried out using the package FRANK which we have developed in the programming language MATLAB. The idea has been to keep all routines as modular as possible, to facilitate possible additions later on. FRANK 1.0 can be downloaded from the authors website.

To keep track of all the information needed, we have defined data structures as follows:

Each Ramsey number R(x,y) is represented as an entry in a matrix of structures, where the structures have the following fields.

x,y: x and y coordinates

ub: upper bound; the currently best known value for an upper bound of $R(x, y)$

lb: lower bound

exact: 1 if the Ramsey number is exact, i.e. if ub=lb, 0 otherwise ubinfo: a text string containing information about which methods were used to calculate upper-bound values

lbinfo: information about lower-bound values

The bounds for the numbers e and E are stored in a similar matrix, which has the following fields:

x,y,n: coordinates

elb: lower bound for e

eub: upper bound for e

exact: 1 if elb=eub

Elb: lower bound for E

Eub: upper bound for E

Exact: 1 if Elb=Eub

info: a text string as for Ramsey numbers

graph: a graph object

minmaxval: the maximum and minimum possible values for valences in the graph, if known

The two last fields are not currently used by any of the routines, but have been left in for possible use in future versions.

Graphs are represented as objects, with the main field of a graph object being its adjacency matrix. (The adjacency matrix for a graph on n vertices is a symmetric n-by-n matrix of 1's and 0's with a 1 in position i,j if there is an edge between vertices i and j.)

Calculations are performed in two stages; the user first creates the two storage matrices above (which are then stored as global variables), and can then apply the various theorems given above or change values manually. When the matrices are initialized, the trivial values of R, e and E given in section 4.4 are added. The functions *updatee* and *updater* implement the recursions described in chapter 4, and perform checks to see if any new exact values have been found by upper and lower bounds coinciding, or, in the case of e-numbers, if any new values have become ∞ by for example the lower bound surpassing the upper bound. The theorems in section 4.4 are also implemented in the code, and are typically used together with the update routines. It is also possible to change values manually, and for convenience some of the published bounds for e have been collected in files so that these can be added.

At least one new bound can be obtained immediately using these routines: The previously published bound $R(3, 15) \leq 89$ is sharpened to 88.

7. Appendix A: Programming routines

This appendix contains a quick reference guide for FRANK version 1.0, and a sample session to show how the software can be used.

7.1. **Quick reference.** This section contains all the help files for FRANK 1.0.

```
AL Lesser values for elbnumbers

   AL manually updates two values for lower
   bounds for e, from "Theoretical and
   computational aspects of Ramsey Theory''.
```

```
   Other values which may be influenced
   are not updated, use UPDATEE.


DISPE display an e number structure

   DISPE(X,Y,N) displays the entry
   in position (X,Y,N) in the global
   variable ETABLE


DISPR display a Ramsey number structure

   DISPR(X,Y) displays the entry
   in position (X,Y) in the global
   variable RTABLE


EELBTABLE Lower bounds for E numbers

   T=EELBTABLE(X) extracts a y-by-n table
   of lower bounds for E numbers E(X,y,n)
   from the global variable ETABLE


EEUBTABLE Upper bounds for E numbers

   T=EEUBTABLE(X) extracts a y-by-n table
   of upper bounds for E numbers E(X,y,n)
   from the global variable ETABLE


EEXETABLE Exact values of E numbers

   T=EEXETABLE(X) extracts a y-by-n table
   of exact values for E numbers E(X,y,n)
   from the global variable ETABLE


ELBTABLE Lower bounds for e numbers

   T=ELBTABLE(X) extracts a y-by-n table
   of lower bounds for e numbers e(X,y,n)
   from the global variable ETABLE


ENUMBERTABLE create global variable RTABLE

   ENUMBERTABLE(xmax,ymax,nmax) creates
```

```
   the global variable ETABLE:
   an xmax-by-ymax-by-nmax matrix of structures
   of e-numbers and E-numbers for two-color
   classical Ramsey numbers.
   The structures have twelve fields:
   et.x, et.y and et.n are the parameters e(x,y,n).
   et.elb is the largest known lower bound for e.
   et.eub is the upper bound for e.
   et.Elb and et.Eub are lower and upper bounds for E.
   et.exact is true if elb=eub.
   et.Exact is true if Elb=Eub.
   et.info is a text string
   with information on how the values
   were calculated.
   et.graph is a graph object
   et.minmaxval is a vector containing the minimum
   and maximum possible valences for a realiser.


EUBTABLE Upper bounds for e numbers

   T=EUBTABLE(X) extracts a y-by-n table
   of upper bounds for e numbers e(X,y,n)
   from the global variable ETABLE


EXETABLE Exact table of e numbers

   T=EXETABLE(X) extracts a y-by-n table
   of exact values for e numbers e(X,y,n)
   from the global variable ETABLE


EXRTABLE Exact table of ramsey numbers

   T=EXRTABLE creates a table of exact bounds
   for traditional two-color Ramsey numbers
   R(x,y) from the global variable RTABLE


GR Grinstead-Roberts values for e-numbers

   GR manually updates values for lower
   bounds for e, Grinstead-Roberts' 1988
   paper. Other values which may be influenced
   are not updated, use UPDATEE.


GY Graver-Yackel values for elbnumbers
```

```
   GY manually updates values for lower
   bounds for e, from Graver-Yackel's 1968
   paper. Other values which may be influenced
   are not updated, use UPDATEE.


JB Backelin values for elbnumbers

   JB manually updates values for lower
   bounds for e, from Backelin's 2000
   manuscript. Other values which may be
   influenced are not updated, use UPDATEE.


LORTABLE Lower table of Ramsey numbers

   T=LORTABLE extracts a table of lower
   bounds for Ramsey numbers R(x,y)
   from the global variable RTABLE


MANUALELB manually update a value of elb

   MANUALELB(X,Y,N,ELB,INFO) sets
   ETABLE(X,Y,N).elb to ELB and adds
   the string INFO to ETABLE(X,Y,N).info


MANUALR manually change a value in the ramsey table RTABLE

   MANUALR(X,Y,N) changes the value of
   R(X,Y) (AND R(Y,X)!) to be exactly N
   MANUALR(X,Y,LB,UB) changes the lower
   and upper bounds to LB and UB.
   The exact field and the info-fields are updated.


RAMEYTABLE create global variable RTABLE

   RAMSEYTABLE(xmax,ymax) creates
   the global variable RTABLE:
   an xmax-by-ymax matrix of structures
   of two-color classical Ramsey numbers.
   The structrures have seven fields:
   rt.x and rt.y are the parameters R(x,y),
   rt.lb is the largest known lower bound,
   rt.ub is the upper bound, and rt.exact is true if
   lb=ub. rt.ubinfo and rt.lbinfo are text strings
```

with information on how the values
were calculated.

RK Radziszowski-Kreher values for elbnumbers

RK manually updates values for lower
bounds for e, from Radziszowski and
Kreher's 1988 paper. Other values which
may be influenced are not updated, use UPDATEE.

THEOREM7 exact values for enumbers

THEOREM7 uses theorem 7 from "Theoretical
and computational aspects of Ramsey Theory"
to calculate exact values of e(3,i+1,n).

THEOREM8 lower bounds for enumbers

THEOREM8 uses theorem 8 from ''Theoretical
and computational aspects of Ramsey Theory'',
which is from Radzisowski/Krehers 1991
paper, and an improvement proved by Backelin

THEOREM9 lower bounds for enumbers

THEOREM9 uses Backelins as yet unpublished
theorem, theorem 9 from "Theoretical and
Computational aspects of Ramsey Theory'' to
improve lower bounds for e-numbers.

TURAN upper bounds for E-numbers

TURAN uses Turan's theorem to improve
values of Eub in the global variable
ETABLE

UPDATEE Update the global enumbertable ETABLE

UPDATEE uses several recursive techniques
to improve bounds for e- and E-numbers.
theorem 4 and the delta inequality
are used to calculate lower bounds for e.
The bounds for E are adjusted so that

```
they agree with the bounds for e, for
example if elb=n then we know that
Elb must be greater than or equal to n.
Updatee also checks whether any new values
have become exact by upper and lower bounds
coinciding, and whether any values have
lower bounds that are greater than upper
bounds, in which case the number cannot
exist and is set to Inf.
```

```
UPDATER update the global ramsey table

   UPDATER performs a number of simple checks
   to improve bounds: First RTABLE is compared
   to ETABLE to see if any new bounds have been

   found, then all values are tested with the simplest
   upper- and lower bound recursions.
   Finally, a check is performed to see if any
   new exact values have been produced.
```

```
UPRTABLE upper table of Ramsey numbers

   T=UPRTABLE extracts a table of upper
   bounds for Ramsey numbers R(x,y)
   from the global variable RTABLE
```

7.2. **A sample session.** The following is a short example showing how some of the features of FRANK can be used.

We begin by creating a 7-by-7 table of Ramsey values. In its initial state the only values given are the trivial values $R(x, 2) = R(2, y) = y$.

```
» ramseytable(7,7)
```

"Exrtable " extracts a table of the exact Ramsey values currently known from this table.

```
» exrtable
ans =
 0 0 0 0 0 0 0
 0 2 3 4 5 6 7
 0 3 0 0 0 0 0
 0 4 0 0 0 0 0
 0 5 0 0 0 0 0
 0 6 0 0 0 0 0
 0 7 0 0 0 0 0
```

Next we create a 3-by-7-by-23 table of bounds for e and E, and update it

```
» enumbertable(3,7,23)
```

```
» updatee
```

"Elbtable " gives a table of lower bounds for $e(x, y, n)$ for a fixed x, so for $x = 3$ we have

```
» elbtable(3)
ans =
 Inf    0    0    0    0    0  0
 Inf    1    0    0    0    0  0
 Inf  Inf    1    0    0    0  0
 Inf  Inf    2    1    0    0  0
 Inf  Inf    5    2    1    0  0
 Inf  Inf  Inf    3    2    1  0
 Inf  Inf  Inf    6    3    2  1
 Inf  Inf  Inf   10    4    3  2
 Inf  Inf  Inf  Inf    7    4  3
 Inf  Inf  Inf  Inf   10    5  4
 Inf  Inf  Inf  Inf   15    8  5
 Inf  Inf  Inf  Inf   20   11  6
 Inf  Inf  Inf  Inf   26   15  9
 Inf  Inf  Inf  Inf  Inf   19 12
 Inf  Inf  Inf  Inf  Inf   25 15
 Inf  Inf  Inf  Inf  Inf   31 20
 Inf  Inf  Inf  Inf  Inf   38 24
 Inf  Inf  Inf  Inf  Inf   45 29
 Inf  Inf  Inf  Inf  Inf  Inf 35
 Inf  Inf  Inf  Inf  Inf  Inf 42
 Inf  Inf  Inf  Inf  Inf  Inf 49
 Inf  Inf  Inf  Inf  Inf  Inf 57
 Inf  Inf  Inf  Inf  Inf  Inf 66
```

We can also change values manually, for example we know from [1] that $e(3, 6, 18) = \infty$. The last argument to the function *manualelb* is a text string where the user can add a comment to the text string "info":

```
» manualelb(3,6,18,Inf,':  Graver-Yackel')
```

The command "dispe " is used to show all information for a particular e-number. The text string "info" shows a record of how different values were calculated, including the comment added above.

```
» dispe(3,6,18)
ans =
 x:  3
 y:  6
 n:  18
 elb:  Inf
 eub:  Inf
 exact:  1
 Elb:  Inf
 Eub:  Inf
 Exact:  1
```

```
info:  '45:eub>Eub 45:delta 45:elb>Elb Inf:Graver-Yackel etoInf '
graph:  [1x1 graph]
minmaxval:  [0 17]
```

We can now use the information contained in the e-number table to update our Ramsey table, and then display the results (uprtable and lortable are tables of the current best upper and lower bounds)

```
» updater
» exrtable

ans =
 0  0  0  0  0  0  0
 0  2  3  4  5  6  7
 0  3  6  9 14 18  0
 0  4  9  0  0  0  0
 0  5 14  0  0  0  0
 0  6 18  0  0  0  0
 0  7  0  0  0  0  0
» uprtable
ans =
 0  0   0  0   0    0   0
 0  2   3  4   5    6   7
 0  3   6  9  14   18  25
 0  4   9 18  31   49  74
 0  5  14 31  62  111 185
 0  6  18 49 111  222 407
 0  7  25 74 185  407 814
» lortable
ans =
 0  0  0  0  0  0  0
 0  2  3  4  5  6  7
 0  3  6  9 14 18 21
 0  4  9 13 19 24 28
 0  5 14 19 28 36 42
 0  6 18 24 36 42 54
 0  7 21 28 42 54 63
```

8. Appendix B: table of $e(3, y, n)$

The following table contains the best values of $e(3, y, n)$ which we have been able to calculate using FRANK. The tablulated values are of three different types:

Values in **boldface** are caluclated values which are immediate consequences of the trivial values mentioned in section 4.4, and by application of theorems 4 through 7.

Values with footnotes are as follows:

0. These two values are given by the calculations in section 5.

1. These values are from Backelins manuscript [1] and were calculated by hand.

3. These values were calculated by Graver and Yackel in [3], by preferring vertices in the possible graphs and examining the number of edges left in H_2.

4. These results are due to Grinstead and Roberts, [4]. They have also used the techniques described in [3] to obtain preliminary bounds for e, and have also developed computer algorithms which, for a given vertex v in an (x, y)-graph G, examine $H_2(v)$ and $H_1(v)$ and list all possible graphs which can be obtained from joining $H_1(v)$ to $H_2(v)$ in such a way that the resulting graph is an (x, y)-graph.

5. McKay and Ke Min showed in [5] that every triangle-free graph on 28 vertices must contain an independent set on 8 vertices, which implies that $e(3, 8, 28) = \infty$. This was done by a similar technique as in [7] which was arrived at independently. Their algorithms work by extending all possible $(7, 22)$, $(7, 21)$ and $(7, 20)$-graphs to attempt to produce an $(8, 28)$-graph if such a graph existed. Since no such graph was found, they conclude that $R(3, 8) = 28$.

7. These values are from Radziszowski and Kreher's 1988 papers, [7] and [8], in which they make use of several computer algorithms to create a catalouge of $(3, y, n, e)$-graphs up to $y = 8$. The basic technique is to use an implementation of the delta inequality to determine if there are any possible solutions, and then to apply an algorithm that they have designed called EXPAND; which given a $(3, k, n - d - 1, e - \deg_2(v))$-graph constructs all $(3, k + 1, n, e)$-graphs which have a vertex v of degree d. Preferring this vertex v thus gives the graph $H_2(v)$ which must be isomorphic to the original graph.

8. These values are given by theorem 8, which Radziszowski and Kreher proved in [9]. In some cases the bounds have been sharpened by using an improvement of this theorem proved by Backelin in [1]; by characterising the possible graphs he proves that the inequality in theorem 8 is strict if $4n > 13k - 4$ and $4n \neq 13k$.

9. These values are given by theorem 9, proved by Backelin in [1].

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